Int. J. Solids Structures, 1971, Vol. 7, pp. 373 to 382. Pergamon Press. Printed in Great Britain

OPTIMAL STRUCTURAL DESIGN FOR GIVEN DEFLECTION IN PRESENCE OF BODY FORCES†

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Abstract—Optimal design of elastic structures for given deflection is discussed under the assumption that some of the loads acting on the structure depend on the design (e.g. weight or inertia forces of elements of structure). A necessary and sufficient condition for local optimality is established, and its use is illustrated by examples.

1. INTRODUCTION

IN MOST of the published work on optimal elastic design for prescribed deflection, body forces, such as gravity, that are proportional to the mass of the element of the structure and hence depend on the as yet unknown design, have been neglected. Icerman [1], however, treated optimal design of sandwich beams of fixed core dimensions for given amplitude of deflection under excitation by a concentrated load, the intensity of which varies harmonically in time. Barnett [2] discussed optimal design of a uniformly accelerating cantilever beam for given tip deflection. His analysis, however, is based on the optimality condition derived in [3] under the assumption that the loads do not depend on the design. The inertia loads of the considered problem clearly violate this assumption.

The present paper is concerned with optimal design of elastic structures for given deflection when some of the loads depend on the design. For the sake of brevity, the general discussion is restricted to beams, but a rod under centrifugal loads is treated as an example. A necessary and sufficient condition for local optimality is established in Section 2, and examples illustrating the use of this condition are discussed in Sections 3–5, with particular attention to numerical procedures.

2. OPTIMALITY CONDITION

Consider a statically determinate or indeterminate beam with the continuously varying bending stiffness $s(x)$, where x denotes distance measured along the beam. Let $u(x)$ and $\bar{u}(x)$ be the actual deflections of this beam under the alternative distributed loads $p(x)$ and $\bar{p}(x)$, and by $u^*(x)$ and $\bar{u}^*(x)$ independent kinematically admissible deflections of the beam. Shield and Prager [4] introduced the concept of the mutual potential energy $U[u^*, \bar{u}^*; s]$ of the beam for the loads $p, \bar{p}$ and the kinematically admissible deflections $u^*, \bar{u}^*$:

$$U[u^*, \bar{u}^*; s] = \frac{1}{2}\left\{\int su^{*\prime\prime}\bar{u}^{*\prime\prime}\,\mathrm{d}x - \int p\bar{u}^*\,\mathrm{d}x - \bar{p}u^*\,\mathrm{d}x\right\}, \tag{2.1}$$

† This research was supported by the U.S. Army Research Office—Durham.

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where the prime denotes differentiation with respect to x. (Note that this energy reduces to the conventional potential energy when $\bar{p} = p$ and $\bar{u}^* = u^*$.) For $u^* = u$ and $\bar{u}^* = \bar{u}$, use of the principle of virtual work furnishes

$$U[u, \bar{u}; s] = -\frac{1}{2}\int su''\bar{u}''\,\mathrm{d}x = -\frac{1}{2}\int p\bar{u}\,\mathrm{d}x = -\frac{1}{2}\int \bar{p}u\,\mathrm{d}x. \tag{2.2}$$

According to the principle of *stationary mutual potential energy* established in [4], the energy $U[u^*, \bar{u}^*; s]$ is stationary in the neighborhood of $u^* = u$, $\bar{u}^* = \bar{u}$. Indeed, variation of U with respect to $\bar{u}^*$ yields

$$\delta_{\bar{u}^*}U = \frac{1}{2}\int [(su^{*\prime\prime})''-p]\delta\bar{u}^*\,\mathrm{d}x, \tag{2.3}$$

which vanishes for $u^* = u$. Similarly, the variation of U with respect to u^* vanishes for $\bar{u}^* = \bar{u}$.

When the principle of mutual potential energy is applied to optimal design for the given deflection $u(x_0) = u_0$ at the cross section x_0, the load $\bar{p}$ is chosen as a unit load at x_0, which may be regarded as the limit, for $\varepsilon \to 0$, of a uniformly distributed load of intensity $1/(2\varepsilon)$ acting on the segment $x_0-\varepsilon \le x \le x_0+\varepsilon$. According to the last term in (2.2), the stationary value of the mutual potential energy then is $-u_0/2$.

Let s and $s^* = s+\delta s$ denote the continuously varying bending stiffness of neighboring designs that satisfy the constraint on the deflection at x_0. Allowing the load $p(x)$ to depend on the design, we set

$$p(x) = q(x)+r(x; s), \qquad p^*(x) = q(x)+r(x; s^*), \tag{2.4}$$

and denote by u, $\bar{u}$ the deflections of the design s under the loads p, $\bar{p}$ and by $u^* = u+\delta u$, $\bar{u}^* = \bar{u}+\delta\bar{u}$ the deflections of the design s^* under the loads p^*, $\bar{p}$. Because

$$u(x_0) = u^*(x_0) = u_0, \tag{2.5}$$

it follows from (2.2) applied to the designs s and s^* that

$$U[u^*, \bar{u}^*; s^*] = U[u, \bar{u}; s]. \tag{2.6}$$

On the other hand, the deflections u, $\bar{u}$ are kinematically admissible for the design s^* and neighboring to its actual deflections under the loads p^*, $\bar{p}$. The principle of stationary mutual potential energy applied to the design s^* therefore furnishes

$$U[u, \bar{u}; s^*] = U[u^*, \bar{u}^*; s^*]. \tag{2.7}$$

Substitution of (2.7) into (2.6) yields

$$U[u, \bar{u}; s^*] - U[u, \bar{u}; s] = 0. \tag{2.8}$$

In view of the definition (2.1) of the mutual potential energy, the expressions (2.4) for the loads, and the constraint (2.5), equation (2.8) is equivalent to

$$\int (s^*-s)u''\bar{u}''\,\mathrm{d}x - \int (r^*-r)\bar{u}\,\mathrm{d}x = 0. \tag{2.9}$$

With $s^* = s+\delta s$ and $r^* = r+\dot{r}\delta s$, where the dot denotes differentiation with respect to s, we finally cast (2.9) into the form

$$\int (u''\bar{u}'' - \dot{r}\bar{u})\delta s \, dx = 0. \tag{2.10}$$

Writing the weight W of the design s as

$$W = \int w(s) \, dx, \tag{2.11}$$

we conclude from (2.10) that

$$\delta W = \int \dot{w}\delta s \, dx = 0 \tag{2.12}$$

if

$$(u''\bar{u}'' - \dot{r}\bar{u})/\dot{w} = \text{const.} \tag{2.13}$$

The condition (2.13) thus is sufficient for the weight of the beam to be stationary with respect to design variations (in the neighborhood of the design s) that satisfy the deflection constraint at x_0. In other words, (2.13) is a *sufficient* condition for local *optimality*.

That this condition is also *necessary* for local optimality may be shown as follows. Denoting the bending moments of the design s under the loads $p, \bar{p}$ by $M, \overline{M}$, we write the constraint on the deflection at x_0 as

$$\int \frac{1}{s} M\overline{M} \, dx - u_0 \leq 0, \tag{2.14}$$

and form the functional

$$F[M, \overline{M}; s] = \int w \, dx + \lambda \left\{ \int \frac{1}{s} M\overline{M} \, dx - u_0 \right\}, \tag{2.15}$$

where the first integral represents the weight of the beam [see (2.11)] and λ is a Lagrangian multiplier. Since M and $\overline{M}$ depend on the design, variation of the design and use of $M = su''$, $\overline{M} = s\bar{u}''$ furnish

$$\delta F = \int (\dot{w} - \lambda u''\bar{u}'')\delta s \, dx + \lambda \int u''\delta\overline{M} \, dx + \lambda \int \bar{u}''\delta M \, dx. \tag{2.16}$$

By the principle of virtual work, the last two integrals in (2.16) equal $\int u\delta\bar{p} \, dx$ and $\int \bar{u}\delta p \, dx$, respectively. The first of these integrals vanishes because the load $\bar{p}$ is not subject to variation, and the second integral may be written as $\int \dot{r}\bar{u}\delta s \, dx$. Since the variation δs may be treated as arbitrary, the relation

$$\dot{w} - \lambda(u''\bar{u}'' - \dot{r}\bar{u}) = 0, \tag{2.17}$$

which is equivalent to (2.13), is a necessary condition for local optimality. Note that λ in (2.17), and hence the constant in (2.13), must be positive if (2.14) is to be fulfilled as equality.

We shall carry the analysis a little further for the case that

$$w = \alpha + \beta s^{(1-m)/m}, \tag{2.18}$$

where α, β, m are constants, and $0 < m < 1$. If the positive factor $(1-m)\beta/m$ in $\dot{w} = [(1-m)\beta/m]s^{(1-2m)/m}$ is absorbed in the positive constant on the right-hand side of (2.13), this optimality condition becomes

$$(u''\bar{u}'' - \dot{r}\bar{u})/s^{(1-2m)/m} = \text{const.}\ (= c^2, \text{say}). \tag{2.19}$$

Multiplying (2.19) by s^2 and simplifying, we obtain

$$c^{2m}s = (M\bar{M} - s^2\dot{r}\bar{u})^m. \tag{2.20}$$

where $M = su''$, $\bar{M} = s\bar{u}''$ are the bending moments of the design s under the loads p, $\bar{p}$.

With the use of (2.20), the deflection constraint may be written as

$$u_0 = \int (M\bar{M}/s)\,\mathrm{d}x = c^{2m}\int M\bar{M}(M\bar{M} - s^2\dot{r}\bar{u})^{-m}\,\mathrm{d}x. \tag{2.21}$$

Elimination of c^{2m} between (2.20) and (2.21) finally yields the relation

$$u_0 s(x) = [M(x)\bar{M}(x) - s^2(x)\dot{r}(x)\bar{u}(x)]^m \int M(\xi)\bar{M}(\xi)[M(\xi)\bar{M}(\xi) - s^2(\xi)\dot{r}(\xi)\bar{u}(\xi)]^{-m}\,\mathrm{d}\xi, \tag{2.22}$$

which is a nonlinear integral equation for the bending stiffness s of the optimal design.

In the important case that $w = \alpha + \beta s$, we have $m = \frac{1}{2}$, and (2.22) may be cast into a form more suitable for numerical work. Squaring (2.22) and bringing the term in $s^2(x)$ on the right to the other side, we find

$$u_0 s(x) = [M(x)\bar{M}(x)]^{\frac{1}{2}}[I^{-2} + \dot{r}(x)\bar{u}(x)/u_0^2]^{-\frac{1}{2}}, \tag{2.23}$$

where I is the integral in (2.22) evaluated for $m = \frac{1}{2}$.

3. CANTILEVER BEAM WITH CONTINUOUSLY VARYING STIFFNESS

Consider a sandwich beam with a light core of fixed breadth B and height $2H$, and identical cover sheets of continuously varying thickness $T/2$. The beam is to be free at the end $x = 0$ and built in at the end $x = l$; it is to be designed for minimum weight of the cover sheets under the constraint that this weight and a given concentrated load Q at $x = 0$ should produce the prescribed deflection $u(0) = u_0$.

If the specific weight of the cover sheets is denoted by γ, we have $r = \gamma T$ and $s = EH^2T$, where E is Young's modulus for the cover sheets. Accordingly,

$$r = \frac{\gamma}{EH^2}s, \qquad \dot{r} = \frac{\gamma}{EH^2}. \tag{3.1}$$

Furthermore,

$$\begin{aligned} M(x) &= Qx + \frac{\gamma}{EH^2}\int_0^x s(\xi)(x-\xi)\,\mathrm{d}\xi, \\ \bar{M}(x) &= x. \end{aligned} \tag{3.2}$$

and

$$\bar{u}(x) = \int_x^l [\overline{M}(\xi)/s(\xi)](\xi - x)\,d\xi = \int_x^l [\xi(\xi - x)/s(\xi)]\,d\xi. \tag{3.3}$$

Introducing the dimensionless quantities

$$y = x/l, \qquad \eta = \xi/l, \qquad S = u_0 s/(Ql^3), \qquad \mu = \gamma l^4/(EH^2 u_0), \tag{3.4}$$

we write (2.23) in the form

$$S(y) = \frac{\{y^2 + \mu y \int_0^y S(\eta)(y-\eta)\,d\eta\}^{\frac{1}{2}}}{\{I^{-2} + \mu \int_y^1 [\eta(\eta - y)/S(\eta)]\,d\eta\}^{\frac{1}{2}}}, \tag{3.5}$$

where

$$I = \int_0^1 \frac{y^2 + \mu y \int_0^y S(\eta)(y-\eta)\,d\eta}{\{y^2 + \mu y \int_0^y S(\eta)(y-\eta)\,d\eta - \mu S^2(y) \int_y^1 [\eta(\eta-y)/S(\eta)]\,d\eta\}^{\frac{1}{2}}}\,dy. \tag{3.6}$$

For $\mu = 0$, equations (3.5) and (3.6) furnish the optimal stiffness

$$S_0(y) = y/2, \tag{3.7}$$

which corresponds to the sole action of the concentrated load Q. For sufficiently small μ, the optimal bending stiffness may be obtained from (3.5) by a perturbation scheme starting with (3.7). One finds

$$S_1(y) = \frac{y}{2} + \mu \frac{y}{24}(-1 + 3y - y^2) \tag{3.8}$$

to within higher order terms in μ. For greater values of μ, the optimal bending stiffness may be found by the following iterative scheme. Starting from (3.7) or (3.8), we compute $S_{n+1}(y)$ from (3.5) by using $S_n(y)$ on the right side of this equation and in (3.6). The integrals in (3.5) and (3.6) are, of course, evaluated by numerical quadrature. The procedure is repeated until two successive results agree within the desired number of significant digits.

The procedure just described was carried out for $\mu = 1, 2$ and 3. The values of dimensionless stiffness (3.4) could be determined to five significant digits in 5 iterations. The corresponding optimal designs are presented in Fig. 1. It was also found that the approximate solution (3.8) agrees with the iteration result within 3 significant figures for $\mu \leq 1$.

4. CANTILEVER BEAM WITH SEGMENTWISE CONSTANT STIFFNESS

While the optimality condition (2.19) was obtained for a beam of continuously varying stiffness, the same procedure may be applied to a beam of segmentwise constant stiffness. Consider a beam of n segments, and let l_i and s_i be length and bending stiffness of the ith segment and x_i the distance measured along the axis of this segment. Deflections and bending moments of this segment in the two states of loading will be denoted by $u_i(x_i)$, $\bar{u}(x_i)$ and $M_i(x_i)$, $\overline{M}_i(x_i)$ and $r_i(x_i; s_i)$ will be used for the part of distributed loads that depends on the stiffness. An analog to (2.22) for the present case is readily found as

$$u_0 s_i = I_i^m \sum_j I_j^{-m} \int M_j(x_j)\overline{M}_j(x_j)\,dx_j, \tag{4.1}$$

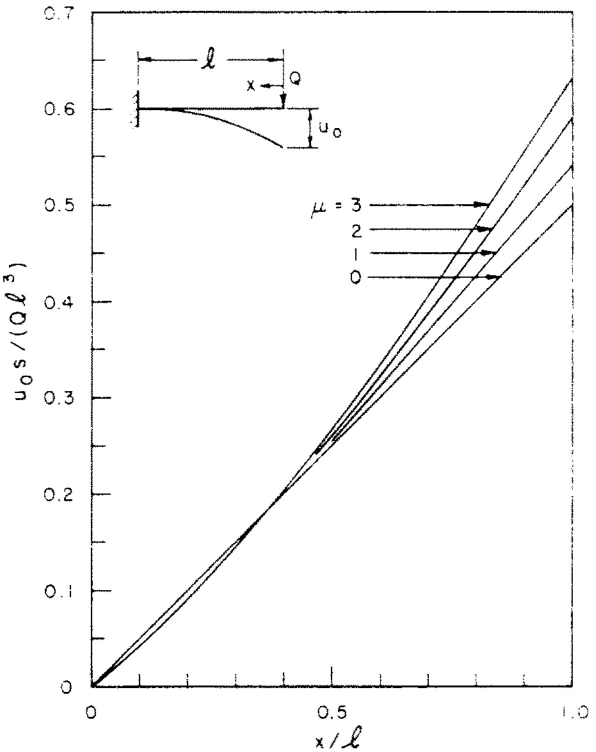

FIG. 1. Optimal designs of cantilever beam with continuously varying stiffness.

where

$$I_i = \frac{1}{l_i}\int [M_i(x_i)\overline{M}(x_i) - s_i^2 \hat{r}_i(x_i)\bar{u}_i(x_i)]\, \mathrm{d}x_i. \tag{4.2}$$

In the important case that $m = \frac{1}{2}$, (4.1) may be cast in the form

$$u_0 s_i = \left[\frac{1}{l_i}\int M_i(x_i)\overline{M}_i(x_i)\,\mathrm{d}x_i\right]\left[I^{-2} + \frac{1}{u_0^2 l_i}\int \hat{r}_i(x_i)\bar{u}_i(x_i)\,\mathrm{d}x_i\right]^{\ldots}. \tag{4.3}$$

where I is the sum in (4.1) evaluated for $m = \frac{1}{2}$. It should be noted that for $n \to \infty$, (4.1) and (4.3) respectively reduce to (2.22) and (2.23).

As a numerical example, the cantilever beam in Section 3 is assumed to consist of two segments with $s = s_1$ for $l_1 \leq x_1 \leq 0$ and $s = s_2$ for $l_1 \leq x_2 \leq l$. Accordingly, (3.1) become

$$r = \frac{\ldots}{EH^2} s_i, \qquad \hat{r} = \frac{\ldots}{EH^2} \tag{4.4}$$

Furthermore,

$$\begin{aligned} M_1(x_1) &= Qx_1 + \frac{\gamma s_1}{2EH^2}x_1^2, \\ M_2(x_2) &= Qx_2 + \frac{\gamma s_1}{2EH^2}l_1(x_2 - l_1) + \frac{\gamma s_2}{2EH^2}(x_2 - l_1)^2, \\ \bar{M}_1(x_1) &= x_1, \\ \bar{M}_2(x_2) &= x_2, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \bar{u}_1(x_1) &= \frac{1}{6s_1}(l_1 - x_1)^2(2l_1 + x_1) + \frac{1}{6s_2}[(l - l_1)^2(2l + l_1) + 3(l^2 - l_1^2)(l_1 - x_1)], \\ \bar{u}_2(x_2) &= \frac{1}{6s_2}(l - x_2)^2(2l + x_2). \end{aligned} \tag{4.6}$$

Introducing the dimensionless quantities

$$\begin{aligned} y = x/l, \qquad \lambda = l_1/l, \qquad S_i = u_o s_i/(Ql^3), \\ z = S_1/S_2, \qquad \mu = \gamma l^4/(EH^2 u_0), \end{aligned} \tag{4.7}$$

we obtain, from (4.3), the following two equations for the unknowns z and S_1:

$$[1 + \lambda + \lambda^2 + \mu C_1(\lambda)]z^2 + \mu C_2(\lambda)z - C_3(\lambda) = 0, \tag{4.8}$$

$$S_1 = [\lambda^3 + (1 - \lambda^3)z][1 - \mu C_4(\lambda)]^{-1}, \tag{4.9}$$

where

$$\begin{aligned} C_1(\lambda) &= \tfrac{1}{24}\{2(4 + \lambda + \lambda^2)[\lambda^4 + (1 - \lambda)(1 - \lambda^3)] - (1 + \lambda + \lambda^2)[3\lambda^4 + (1 - \lambda)^3(3 + \lambda)]\}, \\ C_2(\lambda) &= \tfrac{1}{6}\lambda^3(1 - \lambda)(4 + \lambda + \lambda^2), \\ C_3(\lambda) &= \lambda^2\{1 - \tfrac{1}{24}[3\lambda^4 + (1 - \lambda)^3(3 + \lambda)]\}, \\ C_4(\lambda) &= \tfrac{1}{8}[3\lambda^4 + (1 - \lambda)^3(3 + \lambda) + 2\lambda(1 - \lambda)(4 + \lambda + \lambda^2)]. \end{aligned} \tag{4.10}$$

With the positive root z obtained from (4.8), S_1 is given by (4.9) and S_2 by S_1/z. For given μ, the values of S_1 and S_2 so obtained also depend on λ which must be between 0 and 1. The calculation was carried out for $\mu = 0, 1, 2$ and 3, and the corresponding optimal designs are presented in Fig. 2 as functions of λ. For each μ, there exists a value $\lambda = \lambda_c$ that yields the minimum weight of the optimal beam. It is noted that for the values of μ considered here, the values of λ_c are between 0.424 and 0.442 where the former value corresponds to $\mu = 0$ and the latter to $\mu = 3$.

5. ROTATING ROD

To illustrate an optimal design in which the body force is essential, we consider a rod of length that carries a concentrated mass Q at $x = l$ and rotates at constant angular velocity ω about an axis through $x = 0$ that is perpendicular to the rod. The rod is to be designed for

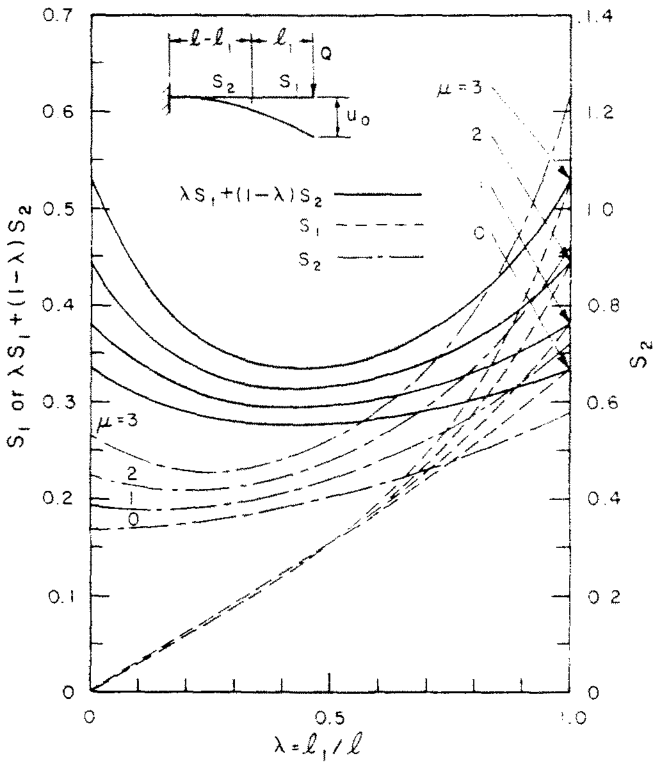

FIG. 2. Optimal designs of cantilever beam with segmentwise constant stiffness.

minimum weight under the constraint that the axial displacement at $x = l$ is not to exceed the given value u_0. The optimal design is given by (2.22) where s is now axial stiffness. M and $\overline{M}$ are, respectively, the axial forces due to the inertia loads and a unit axial load applied at $x = l$ and u and $\bar{u}$ are the corresponding axial displacements. Since the weight per unit length of the rod is proportional to the axial stiffness, we shall set $m = \frac{1}{2}$ in (2.22) and hence obtain (2.23).

If the specific mass of the rod is denoted by ρ, we have $\gamma = \rho\omega^2 xA$ and $s = EA$, where A is the cross-sectional area of the rod. Accordingly,

$$r = \frac{\rho\omega^2}{E} xs, \qquad \dot{r} = \frac{\rho\omega^2}{E} x. \tag{5.1}$$

Furthermore,

$$M(x) = Ql\omega^2 + \int_x^l \frac{\rho\omega^2}{E} \xi s(\xi)\,\mathrm{d}\xi, \qquad \overline{M}(x) = 1, \tag{5.2}$$

and

$$\bar{u}(x) = \int_0^x s^{-1}(\xi)\,\mathrm{d}\xi. \tag{5.3}$$

Introducing the dimensionless quantities

$$y = x/l, \qquad \eta = \xi/l, \qquad S = u_0 s/(Q\omega^2 l^2), \qquad \mu = \rho\omega^2 l^3/(Eu_0), \tag{5.4}$$

we write (2.23) in the form

$$S(y) = \frac{[1+\mu \int_y^1 S(\eta)\eta \, d\eta]^{\frac{1}{2}}}{[I^{-2}+\mu y \int_0^y S^{-1}(\eta) \, d\eta]^{\frac{1}{2}}}, \tag{5.5}$$

where

$$I = \int_0^1 \frac{1+\mu \int_y^1 S(\eta)\eta \, d\eta}{[1+\mu \int_y^1 S(\eta)\eta \, d\eta - \mu y S^2(y) \int_0^y S^{-1}(\eta) \, d\eta]^{\frac{1}{2}}} dy. \tag{5.6}$$

For $\mu = 0$, equations (5.5) and (5.6) furnish the optimal stiffness

$$S_0(y) = 1 \tag{5.7}$$

which corresponds to the sole presence of the concentrated mass Q, the mass of the rod being neglected. For sufficiently small μ, the optimal axial stiffness may be obtained from (5.5) by a perturbation scheme. One finds

$$S_1(y) = 1+\frac{\mu}{12}(7-9y^2) \tag{5.8}$$

to within higher order terms in μ. For greater values of μ, the optimal axial stiffness may be found from (5.5) by an iterative scheme similar to that of Section 3. However, to assure rapid convergence, (5.8) rather than (5.7) is preferred for starting the iteration procedure.

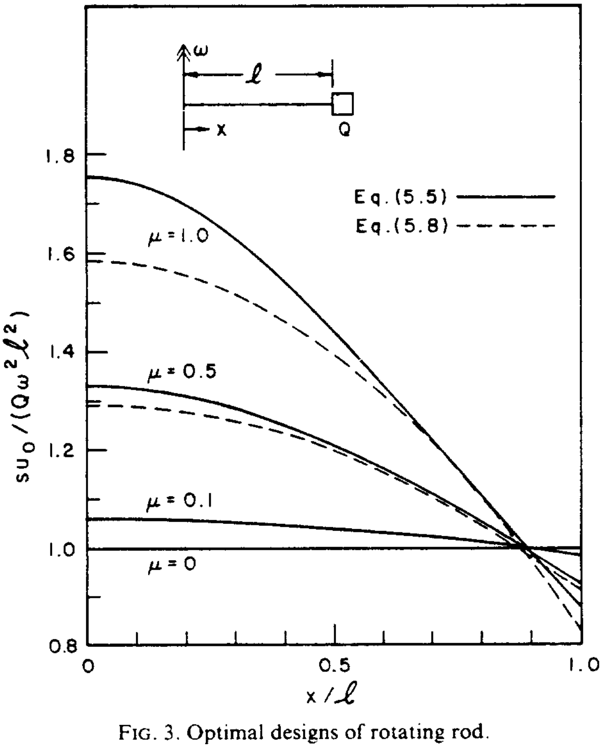

FIG. 3. Optimal designs of rotating rod.

The iteration procedure was carried out for $\mu = 0{\cdot}1$, $0{\cdot}5$ and 1. The axial stiffness could be determined to five significant digits in 6 iterations starting from (5.8) while the same accuracy required 10 iterations starting from (5.7). The corresponding optimal designs are represented in Fig. 3. For comparison the approximate solutions obtained from (5.8) are drawn in Fig. 3 by dashed lines. As is to be expected in the present case, the optimal designs are more sensitive to the value of μ, which is a dimensionless parameter indicating the relative importance of the body force, than were the static cases in Sections 3 and 4.

Acknowledgment The author is indebted to Professor W. Prager for advice in the course of this work

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(*Received* 19 *January* 1970)

Абстракт—Исследуется оптимальный расчет упругих конструкций для заданного прогиба, при предположении, что некоторые из нагрузок, действующих на конструкцию, зависят от проекта конструкции (например вес или инерционные силы элементов конструкции). Устанавливаются необходимое и достаточное условия для локальной оптимальности и иллюстрируется примером их использование.